è-Rolling of a Planar Curve Along Itself and a Connection Between Parabolas and Hyperbolas

by

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ABSTRACT: Let $C_0 : x = f(s), y = g(s), z = 0$, $a \le s \le b$, be a parameterized curve in the *xy*-plane, where *f* and *g* are functions for which $f'(t)g''(t) - g'(t)f''(t) > 0$ (i.e., viewed from the positive *z*-axis, the graph of C_0 turns to the left). Let $P = \big(f(t), g(t), 0\big)$ be a point on C_0. (Of course, as the value of *t* changes, the point *P* moves along the graph of C_0. In this paper, I refer to *s* as the **object parameter** and *t* as the **parameter of animation**.) Let C_θ be the curve in 3-space which has the following properties:

1. C_θ is obtained by a rigid transformation of C_0,
2. C_θ and C_0 share a **tangent vector v** at the point *P*, which lies in the *xy*-plane, and
3. C_θ lies in a plane Π whose **normal vector** makes an angle θ with that of the *xy*-plane, $0 < \theta < 2\pi$.

Then as *P* moves along C_0, the curve C_θ will roll along the graph of C_0; I call the type of rolling envisioned here "the **è - rolling** of C_θ along C_0." In the first part of this paper, I obtain the parametric equations for animating the **è - rolling** of C_θ along C_0 **(Theorem 1).** In the second part, I apply the result to the special case:

C_0 is the **parabola:** $x = s, y = \dfrac{s^2}{4p}, z = 0$, where $p > 0$ is a constant.

I then find the trajectory of focus of the rolling curve C_θ, and find a surprising connection between parabolas and hyperbolas **(Theorem 2).** In the third and final part, I state additional results that can be obtained when (with $\theta = \pi$) the theorems and techniques discussed in this paper are applied to ellipses **(Theorem 3).**

Part I: *Parameterizing the* θ*-rolling of* C_θ *along* C_0.

Theorem 1: Let $C_0 : x = f(s), y = g(s), z = 0$, $a \le s \le b$, be a parameterized curve in the *xy*-plane, where *f* and *g* are functions for which $f'(t)g''(t) - g'(t)f''(t) > 0$ (i.e., viewed from the positive *z*-axis, the graph of C_0 turns to the left). Let $P = \left(f(t), g(t), 0\right)$ be a point on C_0. Then, for every *given* value of θ, $0 < \theta < 2\pi$, parametric equations

$$X = X(s,t),\ Y = Y(s,t),\ Z = Z(s,t),$$

where $a \le s, t \le b$, can be found for animating the θ-rolling of C_θ along C_0.

Proof: To find parametric equations for animating the θ-rolling of C_θ along C_0, we proceed in steps. **Homogeneous matrices** and **homogeneous coordinates** are used to facilitate the required computations, as they allow one to represent *translations* of curves in 3-space in terms of matrix multiplication.

Step 1: Define C_0**.** As a 4x1 column matrix, the homogeneous coordinates of C_0 are

$$C_0 = \begin{bmatrix} f(s) \\ g(s) \\ 0 \\ 1 \end{bmatrix}. \qquad \textbf{(1)}$$

Step 2: Translate C_0 so that *P* lies at the origin. Let *T1* be the **translation matrix** that does this. Then

$$T1 = \begin{bmatrix} 1 & 0 & 0 & -f(t) \\ 0 & 1 & 0 & -g(t) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \qquad \textbf{(2)}$$

Step 3: Let α be the angle that the **tangent vector v** at *P* makes with the *positive x*-axis. **Rotate the translated curve around the *z*-axis through the angle $-\alpha$.** Let *R1* be the **rotation matrix** that does this. Then

$$R1 = \begin{bmatrix} \dfrac{f'(t)}{\sqrt{[f'(t)]^2+[g'(t)]^2}} & \dfrac{g'(t)}{\sqrt{[f'(t)]^2+[g'(t)]^2}} & 0 & 0 \\ \dfrac{-g'(t)}{\sqrt{[f'(t)]^2+[g'(t)]^2}} & \dfrac{f'(t)}{\sqrt{[f'(t)]^2+[g'(t)]^2}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \qquad \textbf{(3)}$$

Step 4: Now rotate the curve around the *x*-axis through an angle θ $(0 < \theta < 2\pi)$. Let *R2* be the rotation matrix that does this. Then

$$R2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \qquad \textbf{(4)}$$

Step 5: Now rotate the curve around the *z*-axis through the angle α. If *R3* is the matrix that does this, then $R3 = R1^{-1}$:

$$R3=\begin{bmatrix}\frac{f'(t)}{\sqrt{[f'(t)]^2+[g'(t)]^2}} & \frac{-g'(t)}{\sqrt{[f'(t)]^2+[g'(t)]^2}} & 0 & 0\\ \frac{g'(t)}{\sqrt{[f'(t)]^2+[g'(t)]^2}} & \frac{f'(t)}{\sqrt{[f'(t)]^2+[g'(t)]^2}} & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}. \qquad \textbf{(5)}$$

Step 6: Translate the curve so that the point at the origin *moves back to P.* If *T2* is the matrix that does this, then $T2 = T1^{-1}$:

$$T2=\begin{bmatrix}1 & 0 & 0 & f(t)\\ 0 & 1 & 0 & g(t)\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}. \qquad \textbf{(6)}$$

Step 7: Finally, for a *given* value for θ ($0<\theta<2\pi$), parametric equations (*X, Y, Z*) for animating the θ – rolling of C_θ along C_0 are given by the *first three rows* of the 4x1 matrix C_θ, where

$$C_\theta = T2\cdot R3\cdot R2\cdot R1\cdot T1\cdot C_0. \qquad \textbf{(7)}$$

Taking the product in **(7)** (I used Maple 10 to assist in this) and letting *X, Y* and *Z* be defined by its first, second and third rows, respectively, we find that, for any *given* value of θ, $0<\theta<2\pi$,

$$X(s,t)=\frac{1}{[f'(t)]^2+[g'(t)]^2}\cdot\Big\{f(s)[f'(t)]^2+f(s)[g'(t)]^2\cos\theta+f'(t)g'(t)g(s)-f'(t)g'(t)g(s)\cos\theta$$
$$-f(t)[g'(t)]^2\cos\theta-f'(t)g'(t)g(t)+f'(t)g'(t)g(t)\cos\theta+f(t)[g'(t)]^2\Big\},$$

$$Y(s,t) = \frac{1}{\left[f'(t)\right]^2 + \left[g'(t)\right]^2} \cdot \Big\{ f'(t)g'(t)f(s) - f'(t)g'(t)f(s)\cos\theta + g(s)\left[g'(t)\right]^2 + g(s)\left[f'(t)\right]^2 \cos\theta$$

$$- f'(t)g'(t)f(t) + f'(t)g'(t)f(t)\cos\theta - g(t)\left[f'(t)\right]^2 \cos\theta + g(t)\left[f'(t)\right]^2 \Big\}$$

and

$$Z(s,t) = \frac{\sin\theta\left[f'(t)g(s) - g'(t)f(s) + g'(t)f(t) - f'(t)g(t)\right]}{\sqrt{\left[f'(t)\right]^2 + \left[g'(t)\right]^2}},$$

where $a \le s,\ t \le b$. This completes the proof of **Theorem 1.**

Part II: *The special case: C_0 is a parabola.*

We now apply **Theorem 1** to the special case where C_0 is the **parabola:** $x^2 = 4py$ and $z = 0$, where $p > 0$ is constant. Parametric equations for C_0 are

$$x = f(s) = s,\ y = g(s) = \frac{s^2}{4p},\ z = 0, \qquad \textbf{(8)}$$

where $-\infty < s < \infty$. Since $f'(t) = 1$ and $g'(t) = \frac{t}{2p}$, then from **(7),** it can be shown that parametric equations (*X, Y, Z*) for animating the θ – rolling of the parabola C_θ along C_0 are

$$X(s,t) = \frac{8sp^2 + 2st^2\cos\theta + ts^2 - ts^2\cos\theta - t^3\cos\theta + t^3}{2\left(4p^2 + t^2\right)},$$

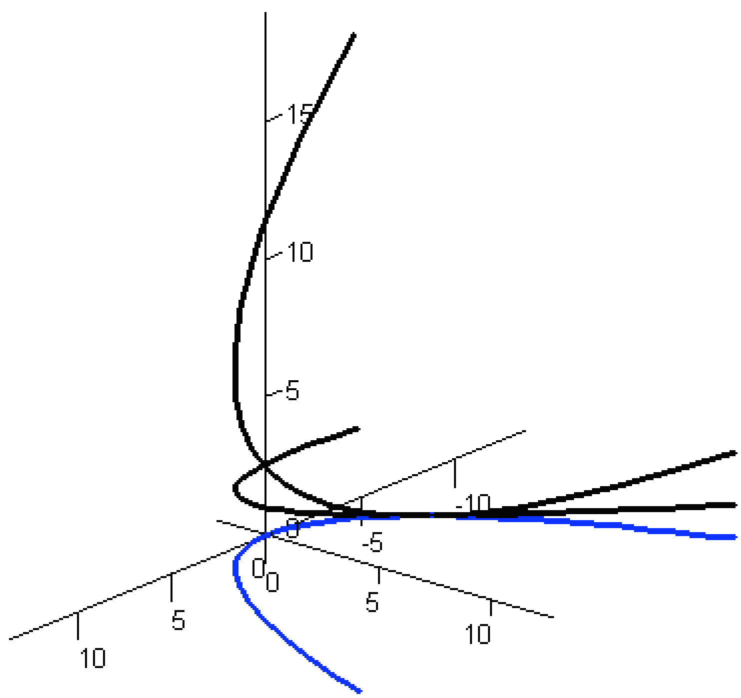
$$Y(s,t) = \frac{8p^2ts - 8p^2ts\cos\theta + s^2t^2 + 4s^2p^2\cos\theta - 4t^2p^2 + 4t^2p^2\cos\theta}{4p\left(4p^2 + t^2\right)}, \qquad \textbf{(9)}$$

and

$$Z(s,t) = \frac{\left(s^2 - 2ts + t^2\right)\sin\theta}{2\sqrt{4p^2 + t^2}},$$

where $-\infty < s, t < \infty$ and $p > 0$ is constant. **Figure 1** below illustrates the result (with $\theta = \dfrac{\pi}{6}$ and $\theta = \dfrac{\pi}{2}$).

FIGURE 1: Pi/6- and Pi/2-Rolling of the Parabola x^2 = 8y Along Itself
(as viewed from the first octant)



Note: If we *specify* a value for *s* (and, hence, a fixed point on C_θ), then as C_θ rolls along C_0, **Equations (9)** provide the parametric equations of the trajectory of that fixed point on the rolling curve. We are now in a position to prove the following theorem.

Theorem 2: Let C_0 be the **parabola:** $x^2 = 4py$ and $z = 0$, where $p > 0$ is constant, and let C_θ be the parabolic curve that $\theta-$rolls along C_θ, where $0 < \theta < 2\pi$ is fixed. If *F* is the **focus** of C_θ, then the trajectory of *F* is

(i) a branch of the **hyperbola:**

$$\frac{z^2}{\sin^2\theta} - \frac{x^2}{(1-\cos\theta)^2} = p^2 \text{ in the plane: } y = p\cos\theta \text{, if } \theta \neq \pi,$$

or

(ii) the **directrix** of C_0, if $\theta = \pi$.

Proof: Let *L* and *R* be the endpoints of the **latus rectum** of the rolling parabola, C_θ. Then

$$F = \frac{1}{2}(L+R)$$

$$= \frac{1}{2}\left(\begin{bmatrix} X(-2p,t) \\ Y(-2p,t) \\ Z(-2p,t) \end{bmatrix} + \begin{bmatrix} X(2p,t) \\ Y(2p,t) \\ Z(2p,t) \end{bmatrix}\right). \qquad \textbf{(10)}$$

Using the formulas in **(9),** it can be shown that **(10)** simplifies to

$$F = \begin{bmatrix} \dfrac{t(1-\cos\theta)}{2} \\ p\cos\theta \\ \dfrac{\sqrt{4p^2+t^2}\sin\theta}{2} \end{bmatrix},$$

and, hence, parametric equations (*x, y, z*) for the trajectory of the focus of the rolling parabola are

$$x = \frac{t\left(1-\cos\theta\right)}{2}, \tag{11}$$

$$y = p\cos\theta , \tag{12}$$

and

$$z = \frac{\sqrt{4p^2+t^2}\sin\theta}{2}, \tag{13}$$

where $-\infty < t < \infty$ and $p > 0$ and $0 < \theta < 2\pi$ are constants. From **(12),** the trajectory of F lies in the plane: $y = p\cos\theta$. We now eliminate the parameter t from **(11)** and **(13).** From **(11),**

$$\frac{t}{2} = \frac{x}{1-\cos\theta} \quad \Rightarrow \quad \frac{t^2}{4} = \frac{x^2}{\left(1-\cos\theta\right)^2}, \tag{14}$$

while from **(13)**, we have

$$z = \sqrt{p^2+\frac{t^2}{4}}\sin\theta \quad \Rightarrow \quad \frac{z^2}{\sin^2\theta} - p^2 = \frac{t^2}{4}, \tag{15}$$

, if $\theta \neq \pi$. From **(14)** and **(15),** we see that the trajectory of F is along a branch of the hyperbola

$$\frac{z^2}{\sin^2\theta} - \frac{x^2}{\left(1-\cos\theta\right)^2} = p^2, \tag{16}$$

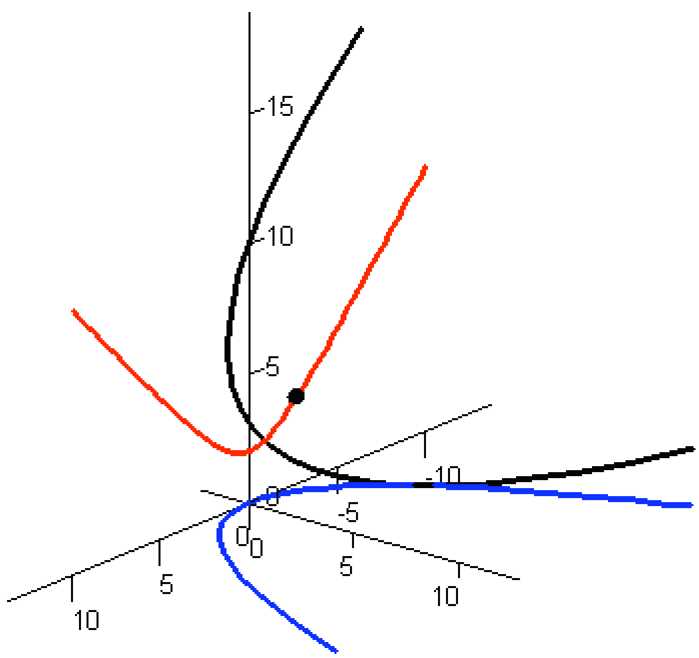
if $\theta \neq \pi$. More specifically, from **(13)** we see that it will be the *top* branch of **(16),** if $0 < \theta < \pi$, and the *bottom* branch, if $\pi < \theta < 2\pi$. On the other hand, if $\theta = \pi$, then from **(11), (12),** and **(13),** the trajectory of the focus F is given by

$$x = t,\ \ y = -p,\ \ z = 0\,, \tag{17}$$

where $-\infty < t < \infty$ and $p > 0$ is constant. The equations in **(17)** are, however, just parametric equations for the directrix of C_0. This completes the proof of **Theorem 2.**

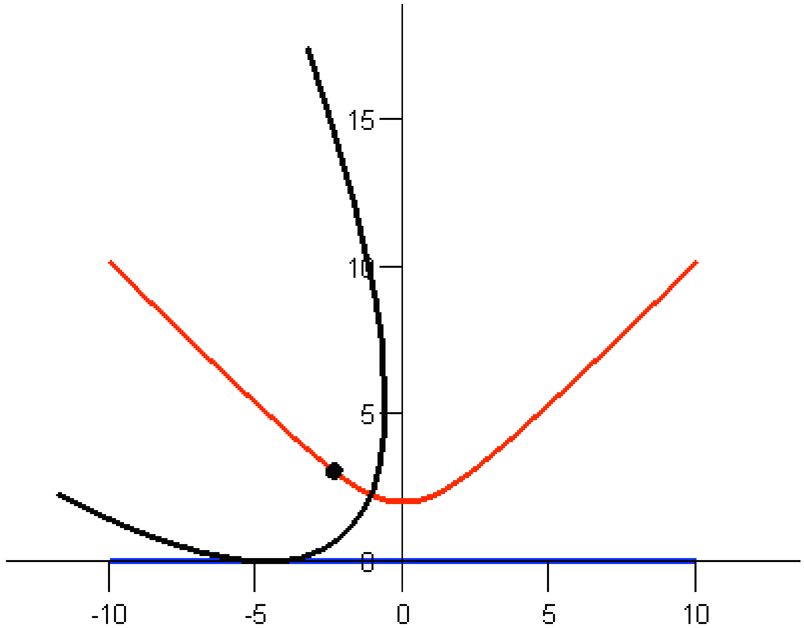
Figure 2 below illustrates the result (with $\theta = \dfrac{\pi}{2}$).

FIGURE 2: Pi/2-Rolling of the Parabola x^2 = 8y Along Itself
Together with the Hyperbolic Trajectory of its Animated Focus
(as viewed from the first octant)



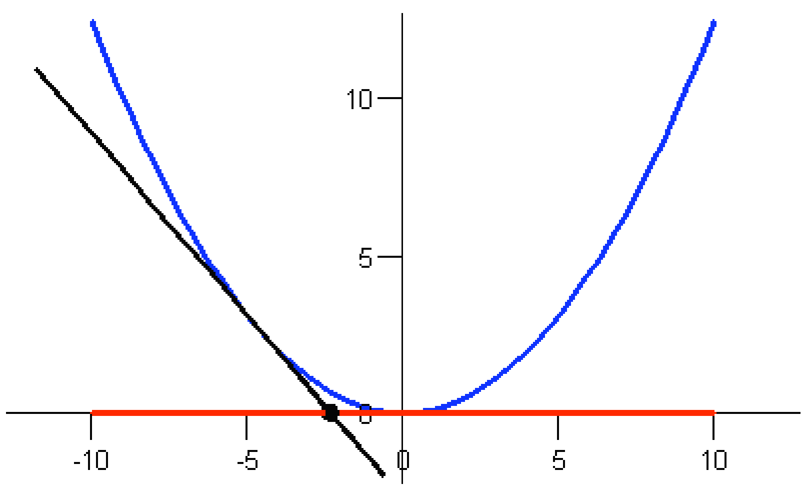
In **Figure 3** below, we view **Figure 2** from a point on the *negative* *y*-axis.

FIGURE 3: Pi/2-Rolling of the Parabola x^2 = 8y Along Itself
Together with the Hyperbolic Trajectory of its Animated Focus
(as viewed from the negative y-axis)



In **Figure 4** below, we view **Figure 2** from a point on the *positive* z-axis.

FIGURE 4: Pi/2-Rolling of the Parabola x^2 = 8y Along Itself
Together with the Hyperbolic Trajectory of its Animated Focus
(as viewed from the positive z-axis)



Note: Not surprisingly, in **Figure 4** the *projection* of the rolling parabola $C_{\pi/2}$ into the *xy*-plane is (part of the) *tangent line* to C_0 at the point *P*. In **Figure 5** and **Figure 6** below, we show the graphs of 17 trajectories of the foci (one being the directrix of C_0).

FIGURE 5: The Hyperbolic Trajectories of 17 Foci
When the Parabola: x^2 = 8y and z = 0 Theta-Rolls Along Itself

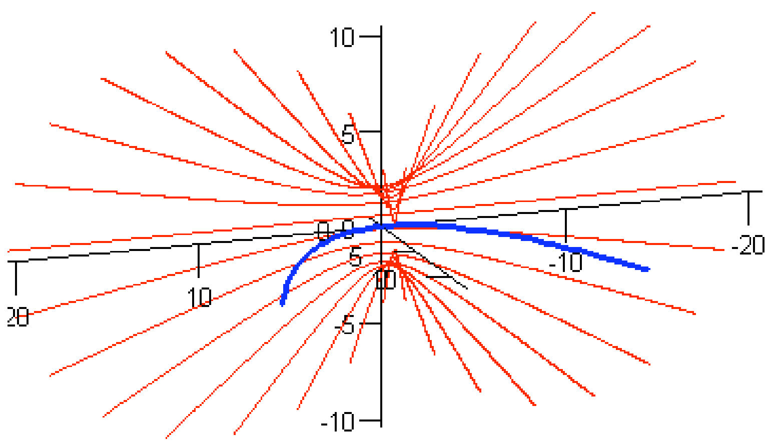
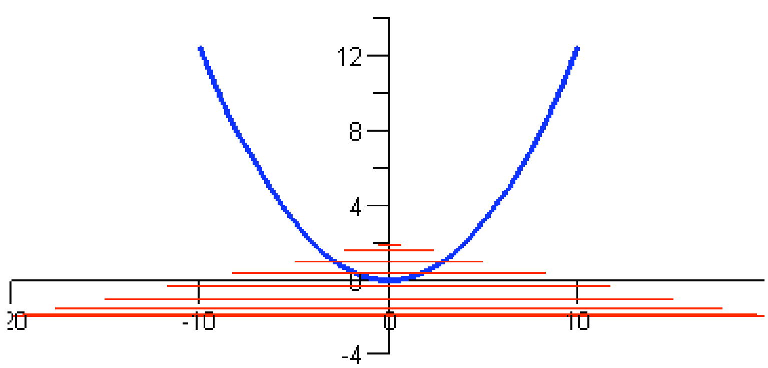


FIGURE 6: The Hyperbolic Trajectories of 17 Foci
When the Parabola: x^2 = 8y and z = 0 Theta-Rolls Along Itself
(as viewed from the positive z-axis)



Part III: *Statement of an additional result*

On February 23, I presented the paper, “Rolling a Parameterized Curve Along its Reflection,” at the ArizMATYC Spring 2007 Meeting, at Glendale Community College. I now recognize that, within the framework of "$\theta -$rolling,” the contents of that paper are merely examples of "$\pi -$Rolling of a Planar Curve Along Itself.” A main result from that paper (merely restated here) is:

Theorem 3: Let C_0 be the *ellipse* having parametric equations

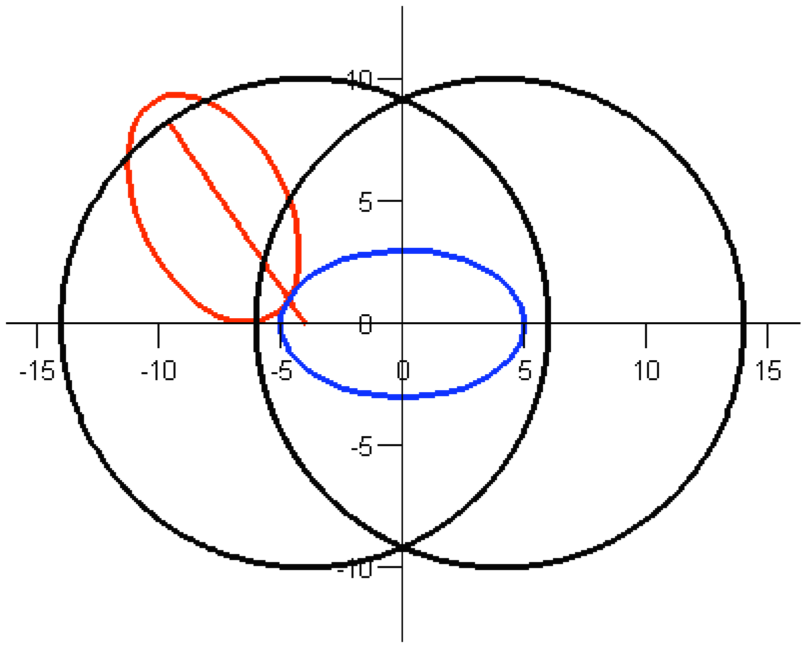
$$x = f(s) = a\cos s,\ y = g(s) = b\sin s,\ z = 0\,,$$

where $0 \le s \le 2\pi$, and *a, b* are constants, with 0 < *b* < *a.* Taking $\theta = \pi$, let C_π be the ellipse that “$\pi -$rolls” around C_0. If $f_1(-c,0)$ and $f_2(c,0)$ are the **foci** of C_0, then the foci of C_π are (initially) at $F_1 = (2a - c, 0)$ and $F_2 = (2a + c, 0)$, respectively. If C_π $\pi -$rolls around C_0 (without sliding), the trajectories of the foci F_1 and F_2 of C_π are **circles** of radius *r* = 2*a*, centered at the foci f_1 and f_2, respectively. Specifically,

$$F_1 \text{ has trajectory: } (x+c)^2 + y^2 = 4a^2 \text{ and } F_2 \text{ has trajectory: } (x-c)^2 + y^2 = 4a^2.$$

Figure 7 below (with *a* = 5 and *b* = 3) is an illustration of **Theorem 3.**

FIGURE 7: An Ellipse (with a = 5 and b = 3) Pi-Rolls Around Itself
(Together with the Circular Trajectories of the Foci)



Final remarks

The theorems proved in this paper began, as do all theorems, as mere conjectures. But, whence *these* conjectures? They entered my mind as a result of posing the following graphical/animation problem: "How could I use my **computer algebra system** (Maple 10) to animate the $\theta-$rolling of one curve along itself?" Having first solved the problem for $\theta = \pi$ ("rolling the reflection of a planar curve along itself"), then for $\theta = \frac{\pi}{2}$ ("orthogonal rolling of a planar curve along itself"), I finally solved the general problem for *any* value of θ, $0 < \theta < 2\pi$ **(Theorem 1).** Having already been surprised at the fact that, when a parabola "orthogonally rolls" along itself, the trajectory of its focus is along the branch of a hyperbola, I decided to investigate what happens in the general case. I was amazed when I discovered that the same was true in general **(Theorem 2)!**